Baum-Connes for quantum groups with torsions

(w/ Adam Skalski (IMPAN))

1. Meyer-Nest formalism for BC

When two spaces have the same cohomology groups, one can expect they might be homotopy equivalent.

The KK-equivalence is the counterpart of the homotopy equiv. in K-theory.

KK is a category

obj: separable C^*-alg

morph: "generalized hom" / homotopy

... counterpart of (stable) homotopy category

KK-equiv

= isom in KK

For $\varphi : A \to B$: $*$-hom

$$M_\varphi := \{ (f, a) \in (C[0,1] \otimes B) \oplus A \mid f(1) = \varphi(a) \}$$

$$\supset C_\varphi := \{ (f, a) \in M_\varphi \mid f(0) = 0 \}$$

$$0 \to C_\varphi \xrightarrow{\iota} M_\varphi \xrightarrow{ev_1} B \to 0$$

exact

$$\begin{array}{c} M_\varphi \\ \wr \ \text{homotopy} \\ A \end{array}$$

Continue this for $\iota : C_\varphi \to M_\varphi$

$$\begin{tikzcd} \to 0 \arrow[r] & E_\iota \arrow[r] \arrow[d, "\wr"] & M_\iota \arrow[r] \arrow[d, "\wr"] & M_\varphi \arrow[r] \arrow[d, "\wr"] & 0 \\ & SB & C_\varphi & A & \end{tikzcd}$$

In this way, we get

$$\cdots \to SA \to SB \to C_\varphi \to A \to B \quad (\circledast)$$

in KK

$\xrightarrow{\text{K-theory}}$ 6-term exact sequence

We write a sequence which is KK-equiv. to $\circledast$ by

$$\begin{tikzcd} C_\varphi \arrow[rr] & & A \arrow[dl] \\ & B \arrow[ul, "\circ"] & \end{tikzcd}$$

and call a (distinguished) triangle

One can define everything in the equivariant setting.

G : (2nd cble) l.c. grp.

KK^G : G-equiv. KK

$\langle \mathcal{CI} \rangle$: full category of proper G-C^*-alg

= "generated" by $\mathrm{Ind}_K^G A$

for $K < G$: cpct subgroup

$\mathcal{CC}$: full subcategory of A s.t. $\mathrm{Res}_G^K A \simeq 0$ in KK^K for any K : cpct. subgroup

Thm. (Meyer–Nest '06)
For $A \in KK^G$, $\exists!$ triangle

$$\begin{tikzcd} P(A) \arrow[rr] & & A \arrow[dl] \\ & N(A) \arrow[ul, "\circ"] & \end{tikzcd}$$

s.t. $P(A) \in \langle \mathcal{CI} \rangle$, $N(A) \in \mathcal{CC}$

Rmk. • $P(A) \to A$ is a variant of the Baum–Connes assembly map.

• If G has Haagerup property

$\mathcal{CC} = 0$ ($\Leftrightarrow P(A) \simeq A$)

↳ We say $\gamma = 1$
strong BC

2. Quantum groups

A compact quantum group (CQG) G is a pair $(C(G), \Delta)$ s.t.

- $C(G)$ is a unital C^*-alg
- $\Delta : C(G) \to C(G) \otimes C(G)$: unital $*$-hom
- $(\Delta \otimes id)\Delta = (id \otimes \Delta)\Delta$
- $\overline{\mathrm{span}}\, \Delta(C(G))(C(G) \otimes 1) = \overline{\mathrm{span}}\, \Delta(C(G))(1 \otimes C(G))$
  $= C(G) \otimes C(G)$

example 1. G : cpct group.

$C(G)$: conti. ftn

$\Delta(f)(s,t) = f(st)$

$\rightsquigarrow (C(G), \Delta)$ is a CQG.

example 2 Γ : discrete group

$C(\hat{\Gamma}) := C^*_\lambda(\Gamma)$

$\Delta(\lambda_s) = \lambda_s \otimes \lambda_s$

$\rightsquigarrow \hat{\Gamma}$ is a CQG

$\exists$ Pontrjagin duality for quantum groups.

CQG is the dual of a DQG

actions, representations can be defined.

By the Baaj-Skandalis duality (= Takesaki-Takai duality in quantum group),

$KK^G \simeq KK^{\hat{G}}$

BC for $\hat{G}$ can be translated in terms of KK^G

This can be done when

$\hat{G}$: torsion-free. (MN '10)

What if $\hat{G}$ has torsion?

Def. $\hat{G}$ is torsion-free if for any finite dimensional ergodic G-C^*-alg is G-Morita equiv to $\mathbb{C}$.

ex. This agrees with the usual notion when $\hat{G}$ is a group.

$$\underset{\Gamma}{\hat{G}}$$

G-C^*-alg $=$ Γ-graded C^*-alg

$\Lambda < \Gamma$: fin. $\rightsquigarrow$ $C^*_\lambda(\Lambda)$: nontrivial G-C^*-alg ex. $SU(2) \twoheadrightarrow SO(3)$

$(\rightsquigarrow \widehat{SO(3)} \subset \widehat{SU(2)})$

$SU(2) \curvearrowright M_2$: inner

$\downarrow$

$SO(3)$ $\curvearrowright$ comes from a proj. rep.

$\rightsquigarrow \widehat{SO(3)}$ is not torsion free even though $\widehat{SU(2)}$ is.

For a subgroup of torsion-free quantum group, one can define the "BC" type $P(A)$ and show

$P(A) \xrightarrow{\simeq} A$ for examples.

(Meyer-Nest '07
Voigt '11
Freslon-Martos '13)

But this does not give $N(A)$

3. crossed product type construction

$G = \hat{\Gamma}$

A, B: G-C^*-alg

($= \Gamma$-graded C^*-alg)

$A \otimes B$ does not admit "product" G-action since

$a \in A_g$. $b \in B_h$

grading of $a \otimes b$

$= gh$? or hg?

Nevertheless,

A: right G-C^*-alg

B: left G-C^*-alg

One can define "the crossed product wrt. the product action"

$A \rtimes G \ltimes B$.

For torsion D,

D^{op} admits a natural right G-action.

$\rightsquigarrow D^{op} \rtimes G \ltimes B$

$\langle \mathcal{C}f \rangle$: cat. gen. by $D \otimes A$ for some D : f.d.

$\mathcal{N}$: A s.t. $D^{op} \rtimes G \ltimes A \sim 0$ in KK

<u>Thm.</u> (A-S)

For $A \in KK^G$

$\exists!$

$$\begin{tikzcd} P \arrow[r] \arrow[dr] & A \arrow[d] \\ & N \end{tikzcd}$$

$P \in \langle \mathcal{C}f \rangle$

$N \in \mathcal{N}$

Def $\langle \mathcal{C}f \rangle$-BC property

$\overset{\text{def}}{\iff}$ $P \to A$: KK-equiv.

$\langle \mathcal{C}f \rangle$-strong-BC property

$\overset{\text{def}}{\iff}$ $P \to A$: KK^G-equiv.

Rmk • This P does NOT coincide with $P(A)$ when $G = \Gamma$.

($\langle \mathcal{C}f \rangle$ is smaller than $\langle \mathcal{C}\mathcal{I} \rangle$)
$\quad \mathbb{N} \quad$ larger $\quad = \mathcal{C}\mathcal{C}$

In particular

We do not know

$\langle \mathcal{C}f \rangle$-strong BC for $G = \{\Gamma\}$

Γ : fin.

Nevertheless $\langle \mathcal{C}f \rangle$-BC is equiv. to usual BC.

4 Applications

① G: CQG with (Cof)-BC

$G \curvearrowright A$

$A \rtimes G$: UCT if

- A: type I or
- A: UCT, $\hat{G}$: torsion-free

② ◦ $C(G)$: QD $\Rightarrow$ $\hat{G}$: amen

◦ $\hat{G}$: amen, (Cof)-BC, of Kac

$\Rightarrow$ $C(G)$: QD.

◦ $C(SU_q(2))$: not QD.